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The Shannon-McMillan Theorem for AF C^*-systems

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The classical Shannon-McMillan Theorem [S] states that an ergodic system has *typical sets* satisfying the asymptotic equipartition property. This theorem demonstrates the significance of the entropy which gives the *size* of the typical sets. Needless to say, entropy is an important notion in statistical mechanics and in a way the asymptotic equipartition property can be seen as the equivalence of ensembles. The asymptotic equipartition property is also important in source coding theory, giving the possible compression rate.

There has recently been great progress in the quantum version of the Shannon-McMillan Theorem [HP2], [NS1], [BKSS], [BS]. In particular, Bjelaković et al. [BKSS] proved Shannon-McMillan theorem for ergodic quantum spin systems. As for the classical Shannon-McMillan Theorem, the Shannon-McMillan Theorem in quantum spin systems is related to the equivalence of ensembles and the quantum source coding in the quantum information theory.

In this talk, we propose a new approach to such results. In the standard proof of the classical Shannon-McMillan Theorem, conditional expectation plays an important role. However, in quantum systems, there is no suitable conditional expectation in general, and classical proof is not applicable there. In [HP2], [NS1], [BKSS], [BS], the analyses were done by reducing the quantum setting to a classical one, and applying the classical Shannon-McMillan Theorem. The reduction is done by use of the ergodic decomposition and approximation of the quantum entropy by classical ones. The quantum mean entropy of a translation invariant state can be approximated by classical mean entropies of its restriction to some abelian subalgebras.

Here, we introduce a direct proof of the quantum Shannon-McMillan theorem, without relying on the classical theory. Our proof is based on the variational principle, which is a well-known thermodynamic property of quantum spin systems. Roughly speaking, the variational principle enables us to estimate *rank* of support projections of ergodic states, in terms of the mean entropy. By virtue of this estimate, we are able to prove quantum Shannon-McMillan theorem directly, without relying on the classical version of it.

Using this argument, we extend the quantum Shannon-McMillan theorem to AF C^*-systems. AF C^*-system is a natural generalization of quantum spin system and its Gibbs structure is studied in [HP1],[GN]. Our proof applies to any dynamical system which admits thermodynamical formalism. In particular, we can apply it to quantum spin systems on $\mathbb{Z}^\nu$-lattice with $\nu \geq 2$.

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